Problem 2.14

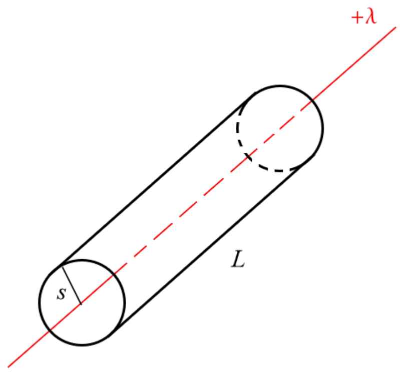
Find the electric field a distance s from an infinitely long straight wire that carries a uniform line charge λ. Compare Eq. 2.9.

Solution

One of the governing equations for the electric field in vacuum is Gauss's law.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

Normally the curl of $\mathbf{E}$ is also necessary to determine $\mathbf{E}$, but because of the cylindrical symmetry, the divergence is sufficient. Integrate both sides over the volume of a (black) coaxial cylindrical Gaussian surface with radius s and length L.



The enclosed charge is the product of the charge density with the length.

$$\begin{aligned}\int_0^L \int_0^{2\pi} \int_0^s \nabla \cdot \mathbf{E}\,(s_0\,ds_0\,d\phi_0\,dz_0) &= \int_0^L \int_0^{2\pi} \int_0^s \frac{\rho}{\epsilon_0}(s_0\,ds_0\,d\phi_0\,dz_0) \\ &= \frac{1}{\epsilon_0}\int_0^L \int_0^{2\pi} \int_0^s \rho\,(s_0\,ds_0\,d\phi_0\,dz_0) \\ &= \frac{1}{\epsilon_0}(\lambda L)\end{aligned}$$

Apply the divergence theorem on the left side.

$$\int_0^L \int_0^{2\pi} (\mathbf{E} \cdot d\mathbf{S}_0)\Big|_{s_0=s} = \frac{\lambda L}{\epsilon_0}$$

Because of the cylindrical symmetry, the electric field is expected to be entirely radial:
$\mathbf{E} = E(s)\hat{\mathbf{s}}$. Note also that the direction of $d\mathbf{S}$ is the outward unit vector to the Gaussian surface.

$$\int_0^L \int_0^{2\pi} [E(s)\hat{\mathbf{s}}_0] \cdot (\hat{\mathbf{s}}_0\,s\,d\phi_0\,dz_0) = \frac{\lambda L}{\epsilon_0}$$

Evaluate the dot product.

$$\int_0^L \int_0^{2\pi} sE(s)\,d\phi_0\,dz_0 = \frac{\lambda L}{\epsilon_0}$$

$E(s)$ is constant on the cylindrical Gaussian surface, so it can be pulled in front of the integral.

$$sE(s)\left(\int_0^{2\pi} d\phi_0\right)\left(\int_0^L dz_0\right) = \frac{\lambda L}{\epsilon_0}$$

Evaluate the integrals.

$$sE(s)(2\pi)(L) = \frac{\lambda L}{\epsilon_0}$$

Solve for $E(s)$.

$$E(s) = \frac{\lambda}{2\pi\epsilon_0 s}$$

Therefore, the electric field around the infinitely long wire is

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}}.$$

This is the same result in Eq. 2.9 but with s instead of z.